SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2011
HIGHER SCHOOL CERTIFICATE ASSESSMENT TASK #2

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 70

- Attempt questions 1 – 3
- All questions are not of equal value.
- Unless otherwise directed give your answers in simplest exact form.

Examiner: *A.M.Gainford*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \ n \neq -1; \ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \ x > 0$

START A NEW BOOKLET

Question 1. (22 marks)

Marks

(a) Evaluate $\left(\dfrac{49}{16}\right)^{-\frac{3}{2}}$, as a common fraction in simplest form. **2**

(b) (i) Express $1\cdot 352$ radians in degrees, correct to the nearest minute. **2**

(ii) Find sin 5, correct to four significant figures.

(c) Differentiate **8**

(i) $3x^2 - 5x + 7$

(ii) $\dfrac{3}{x^4}$

(iii) $\left(x^2 - 1\right)^5$

(iv) $x^2(1-x)^4$

(d) Consider the function $y = 3\sin 2x$. **4**

(i) State the amplitude and period of the function.

(ii) Sketch the graph of the function in the domain $0 \le x \le 2\pi$

(e) (i) Find $\int\left(3x^2 + 4x - 7\right)dx$ **6**

(ii) Evaluate $\int_0^3\left(2x^2 + x\right)dx$

(iii) Find $\int\dfrac{1-x^3}{x^2}\,dx$

START A NEW BOOKLET

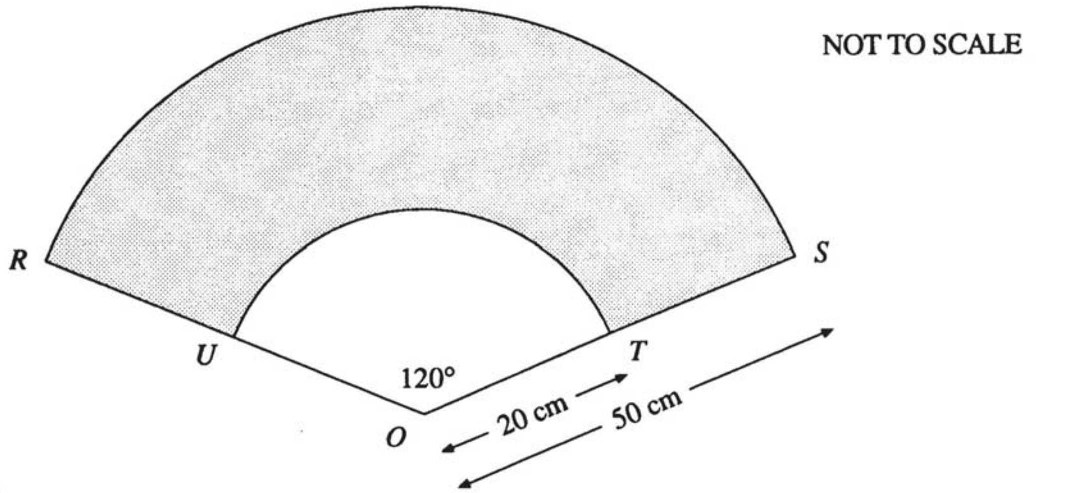
Question 2 (25 marks)

Marks

(a) (i) Find $\log_3 81$. **4**

(ii) Given that $\log_4 9 = 1\cdot585$, correct to 3 decimal places, find $\log_4 144$.

(b) **4**



A car windscreen wiper sweeps out the shape *RSTU*, where *RS* and *UT* are arcs of circles centre *O*. Measurements are as shown in the figure.

(i) Calculate the perimeter of *RSTU*.

(ii) Calculate the area *RSTU*.

(c) **4**

x	1	2	3	4	5
$f(x)$					

(i) Copy and complete the table for $f(x) = \dfrac{3\sqrt{x}}{x+1}$, correct to 4 decimal places.

(ii) Using Simpson's Rule with the above function values find an estimate for $\displaystyle\int_1^5 \frac{3\sqrt{x}}{x+1}\,dx$, correct to 4 significant figures.

(d) Find that part of the domain for which $f(x) = 2x^3 + 3x^2 - 36x + 1$ is a decreasing function. **2**

(e) Given three points $A(0,\sqrt{3})$, $B(3,0)$, $C(2,-\sqrt{3})$: **5**

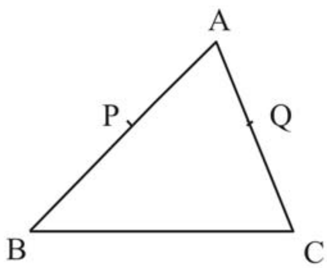
(i) Draw a diagram to represent this situation.

(ii) Show that AB and BC meet at right angles.

(iii) If D is the point (1, 0), show that A, B and C lie on a circle with centre D.

(f) In the adjoining figure P and Q are the midpoints of AB and AC respectively. **3**

Prove that $PQ \parallel BC$, and that PQ is half the length of BC.



(g) The area under the curve $y = 1 - x^2$ from $x = 0$ to $x = 1$ is rotated about the x-axis. Find the volume of the solid of revolution generated. **3**

START A NEW BOOKLET

Question 3 (23 Marks)

Marks

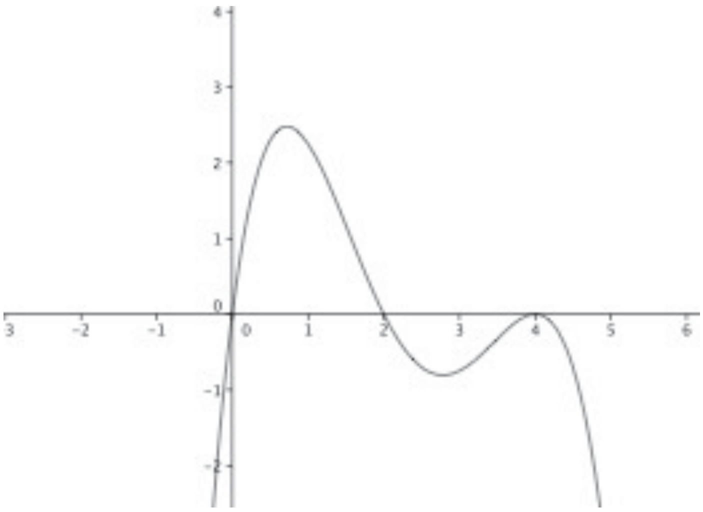
(a) Two identical urns each contain a number of numbered pool balls. Urn A contains three balls numbered 1, 2, 3, whereas Urn B contains five balls numbered 1, 3, 5, 7, 9. **4**

A ball is drawn at random from each urn.

(i) What is the probability that both balls have the same number?

(ii) What is the probability that at least one ball is a 3?

(b) The diagram shows the graph of a certain derivative, $y = f'(x)$. **3**



(i) Copy this diagram to your Writing Booklet.

(ii) On the *same* set of axes, draw a sketch of a possible $f(x)$.

(c) Consider the curves $y = x^2 - 4$ and $y = 2 - x^2$. **4**

(i) Sketch the graphs of the curves on the same axes, and state the x-values of the points of intersection.

(ii) Hence find the area bounded by the two curves, between their points of intersection.

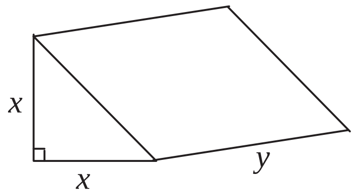
(d) Given the curve $y = x^3 - 6x^2 - 15x$ where $-3 \le x \le 9$. **6**

(i) Find any stationary points, and points of inflexion.

(ii) Sketch the curve, showing its principal features.

(iii) State the maximum and minimum values of y in the domain.

(e) An isosceles right-triangle based prism, with dimensions x cm and y cm (as shown), is to have a volume of 1000 cm^3. **6**



(i) Write equations for the volume (V) and surface area (S) of the figure.

(ii) Show that the surface area $S = x^2 + \dfrac{2000\left(2+\sqrt{2}\right)}{x}$.

(iii) Find the value of x (correct to one decimal place) so that the surface area is a minimum.

This is the end of the paper.

2 unit Year 12 2011

START A NEW BOOKLET

Question 1. (22 marks)

Marks

(a) Evaluate $\left(\frac{49}{16}\right)^{-\frac{3}{2}}$, as a common fraction in simplest form. **2**

$\left(\frac{16}{49}\right)^{\frac{3}{2}} = \frac{4^3}{7^3} = \frac{64}{343}$ 2

(b) (i) Express $1 \cdot 352$ radians in degrees, correct to the nearest minute. **2**

$\pi^c = 180^\circ$

$1^c = \frac{180}{\pi}$

$1.352 \times \frac{180}{\pi} = 77^\circ 28'$ 1

(ii) Find sin 5, correct to four significant figures.

-0.9589 1

(c) Differentiate **8**

(i) $\frac{d}{dx}(3x^2 - 5x + 7) = 6x - 5$ 1

(ii) $\frac{d}{dx}\left(\frac{3}{x^4}\right) = \frac{d}{dx}(3x^{-4}) = -12x^{-5} = \frac{-12}{x^5}$ 2

(iii) $\frac{d}{dx}(x^2 - 1)^5 = 5(x^2-1)^4 \times 2x = 10x(x^2-1)^4$ 2

(iv) $\frac{d}{dx} x^2(1-x)^4 = x^2 \times 4(1-x)^3 \times -1 + (1-x)^4 \times 2x$

$= 2x(1-x)^3[-2x + (1-x)] = 2x(1-x)^3[-3x+1]$ 3

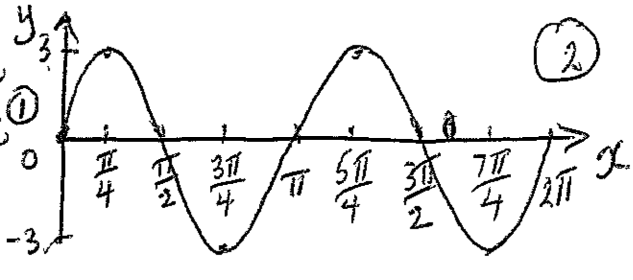
(d) Consider the function $y = 3\sin 2x$. **4**

(i) State the amplitude and period of the function.

amplitude 3 1

period $= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ 1

(ii) Sketch the graph of the function in the domain $0 \le x \le 2\pi$



2

(e) (i) Find $\int (3x^2 + 4x - 7)\,dx = \frac{3x^3}{3} + \frac{4x^2}{2} - 7x + c$

$= x^3 + 2x^2 - 7x + c$ 1

(ii) Evaluate $\int_0^3 (2x^2 + x)\,dx$

$\left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^3$

$= \left(\frac{2}{3} \times 27 + \frac{9}{2}\right) - (0+0)$

$= 18 + 4\frac{1}{2}$

$= 22\frac{1}{2}$ 2

(iii) Find $\int \frac{1-x^3}{x^2}\,dx$

$= \int \frac{1}{x^2} - \frac{x^3}{x^2}$

$= \int x^{-2} - x \; dx$

$= \frac{x^{-1}}{-1} - \frac{x^2}{2} + c$

$= -\frac{1}{x} - \frac{x^2}{2} + c$ 3

Question 2

a.i. $\log_3 81 = 4$ ①

ii $\log_4 144 = \log_4 9 + \log_4 16$ ①
$= 1.585 + \log_4 4^2$
$= 1.585 + 2\log_4 4$ ②
$= 1.585 + 2$
$= 3.585$ ③

b. $120° = 2\pi/3$

i. $P = 2(50-20) + 20 \times 2\pi/3 + 50 \times 2\pi/3$
$= 206.61\text{cm}$ (2dp) ②

ii $A = (\tfrac{1}{2} \times 50^2 \times 2\pi/3) - (\tfrac{1}{2} \times 20^2 \times 2\pi/3)$
$= 2199.11\ \text{cm}^2$ (2dp) ②

c.

i

x	1	2	3	4	5
$f(x)$	1.5	1.4142	1.2990	1.2	1.1180

①

ii $h = \frac{5-1}{4} = \frac{4}{4} = 1$

$$\int_1^5 \frac{3\sqrt{x}}{x+1}\,dx$$

$$\doteqdot \frac{1}{3}\Big[1.5 + 1.1180 + 4(1.4142 + 1.2) + 2(1.2990)\Big]$$

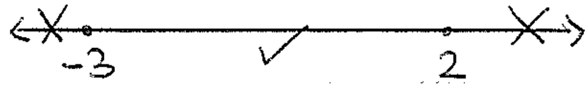
$= 5.224$ (4 sig fig) ③

d. decreasing $\therefore f'(x) < 0$

$f'(x) = 6x^2 + 6x - 36$ ½

St. at $f'(x) = 0$

$6x^2 + 6x - 36 = 0$
$x^2 + x - 6 = 0$
$(x+3)(x-2) = 0$
$\therefore x = -3,\ x = 2$

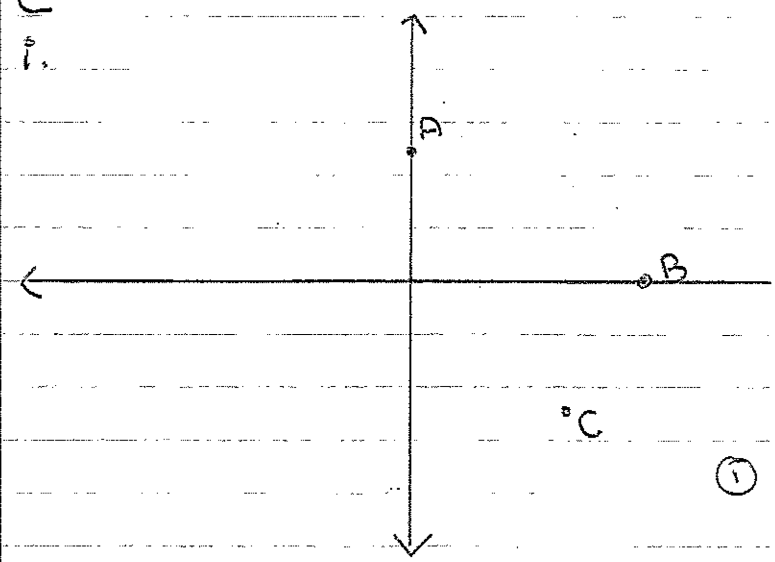


$f'(-4) = 36$ $f'(0) = -36$ $f'(3) = 36$

$\therefore -3 < x < 2$ ②

e

i.



①

ii M of $AB = \frac{\sqrt{3} - 0}{0 - 3}$
$= \frac{-\sqrt{3}}{3}$

M of $BC = \frac{0 + \sqrt{3}}{3 - 2}$
$= \sqrt{3}$

$M_1 \times M_2 = \frac{-\sqrt{3}}{3} \times \sqrt{3}$
$= -3/3$
$= -1$

$\therefore AB \perp BC$ ②

iii d of $BD = \sqrt{(3-1)^2 + (0-0)^2}$
$= \sqrt{2^2}$
$= 2$

$\therefore$ centre $(1, 0)$ radius 2

circle eqn:
$(x-1)^2 + y^2 = 4$

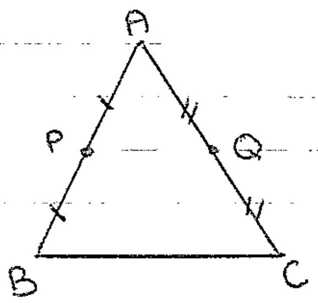
Check A: $(0-1)^2 + (\sqrt{3})^2$
$= 1 + 3$
$= 4$

Check B: $(3-1)^2 + 0^2$
$= 2^2$
$= 4$

Check C: $(2-1)^2 + (-\sqrt{3})^2$
$= 1 + 3$
$= 4$

$\therefore$ A, B & C lie on a circle with centre D. (2)

f.



$\triangle APQ ||| \triangle ABC$ as $\angle A$ is common & $\frac{AP}{AB} = \frac{1}{2} = \frac{AQ}{AC}$.

"Two triangles are similar if there is an equal angle & the sides making this angle are in the same ratio."

$\therefore \angle APQ = \angle ABC$
& $\angle AQP = \angle ACB$
(corresponding $\angle$'s in $|||$ $\triangle$'s)

$\therefore PQ \parallel BC$. (2)

$\frac{PQ}{BC} = \frac{AP}{AB}$ (sides in same ratio)
$\frac{PQ}{BC} = \frac{1}{2}$
$PQ = \frac{1}{2} BC$. (1)

g. $V = \pi \int_0^1 (1 - x^2)^2 \, dx$ (1)
$= \pi \int_0^1 (1 - 2x^2 + x^4) \, dx$
$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1$ (2)
$= \pi \left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 \right]$
$= \frac{8\pi}{15}$. (3)

Solns 2Unit Yr12

$\overline{23}$

3 (a)

A	B
1, 2, 3	1, 3, 5, 7, 9

(i) $P(1,1 \text{ or } 3,3)$

$= \frac{1}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{1}{5}$

$= \frac{1}{15} + \frac{1}{15}$

$= \frac{2}{15}$

2

(ii) $P(\text{at least 1 ball is a } 3)$

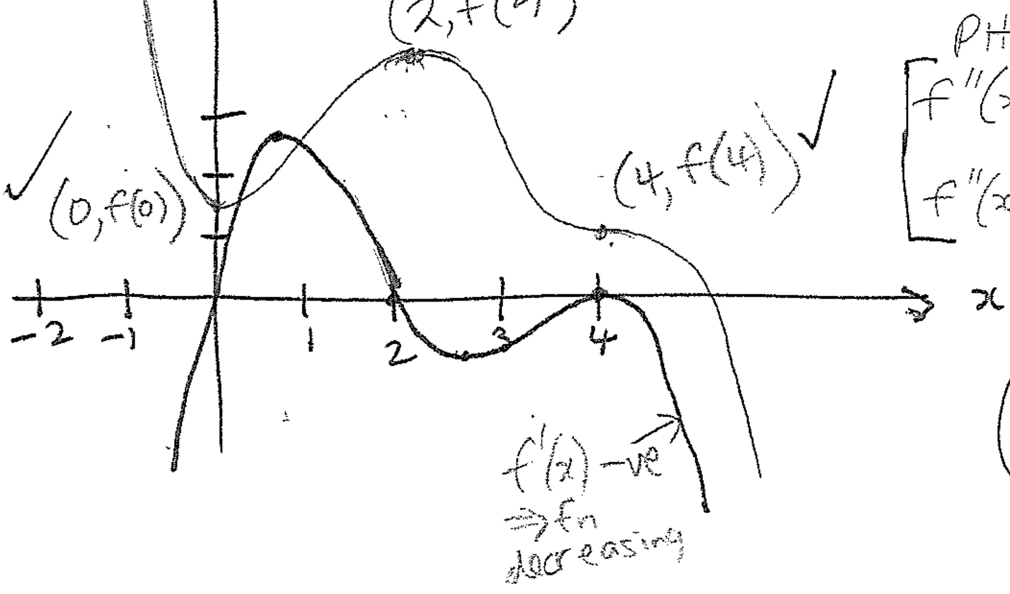
$= P(3,\bar{3} \text{ or } \bar{3},3) \text{ or } 3,3)$ OR $1 - P(\bar{3},\bar{3})$

$= \frac{1}{3}\times\frac{4}{5} + \frac{2}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{1}{5}$ 2 $= 1 - \frac{2}{3}\times\frac{4}{5}$

$= \frac{4}{15} + \frac{2}{15} + \frac{1}{15}$ $= 7/15$

$= \frac{7}{15}$

b)



Min TP at $x = 0$

Max TP at $x = 2$

PHI at $x = 4$

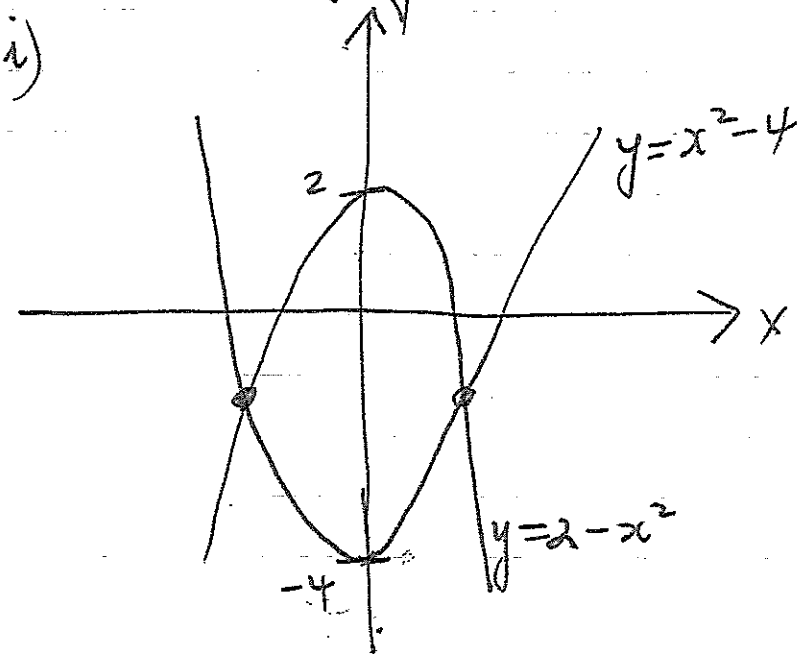
$f''(x) = 0$ at $x \doteqdot .8$
⇒ change in concav

$f''(x) = 0$ at $x \doteqdot 2.9$
⇒ change in concavit

3

3(c) $y = x^2 - 4$ ①
$y = 2 - x^2$ ②

(i)



✓

Points of intersection – solve simultaneously

$\Rightarrow x^2 - 4 = 2 - x^2$

$2x^2 = 6$

$x^2 = 3$

$x = \pm\sqrt{3}$ ✓

2

(ii) $A = \int_{-\sqrt{3}}^{\sqrt{3}} (2-x^2)-(x^2-4)\,dx$ ✓

$= \int_{-\sqrt{3}}^{\sqrt{3}} (-2x^2+6)\,dx$

$= \left[-\frac{2x^3}{3} + 6x\right]_{-\sqrt{3}}^{\sqrt{3}} = \left(\frac{-2\times(+\sqrt{3})^3}{3} + 6\times+\sqrt{3}\right) - \left(\frac{-2\times(-\sqrt{3})^3}{3} + 6\times-\sqrt{3}\right)$

$= \frac{-2\sqrt{3}^3}{3} + 6\sqrt{3} - \frac{2\sqrt{3}^3}{3} + 6\sqrt{3}$ ✓

$A = -4\sqrt{3} + 12\sqrt{3} = 8\sqrt{3}$ sq. units.

2

(d) $y = x^3 - 6x^2 - 15x \quad , \quad -3 \leq x \leq 9$

(i) $y' = 3x^2 - 12x - 15$

$y'' = 6x - 12$

For t.p's $y' = 0 \Rightarrow 3(x^2 - 4x - 5) = 0$ ✓

$3(x-5)(x+1) = 0$

$\Rightarrow \underline{x = 5 \text{ or } -1}$

When $x = 5$, $y = -100 \rightarrow (5, -100)$ ✓

$x = -1$, $y = 8 \rightarrow (-1, 8)$

<u>Type of St. Point</u>

$y''(5) = 30 - 12 > 0 \Rightarrow$ <u>min</u> at $(5, -100)$

$y''(-1) = -6 - 12 < 0 \Rightarrow$ <u>max</u> at $(-1, 8)$

<u>Points of Inflexion</u> When $y'' = 0$

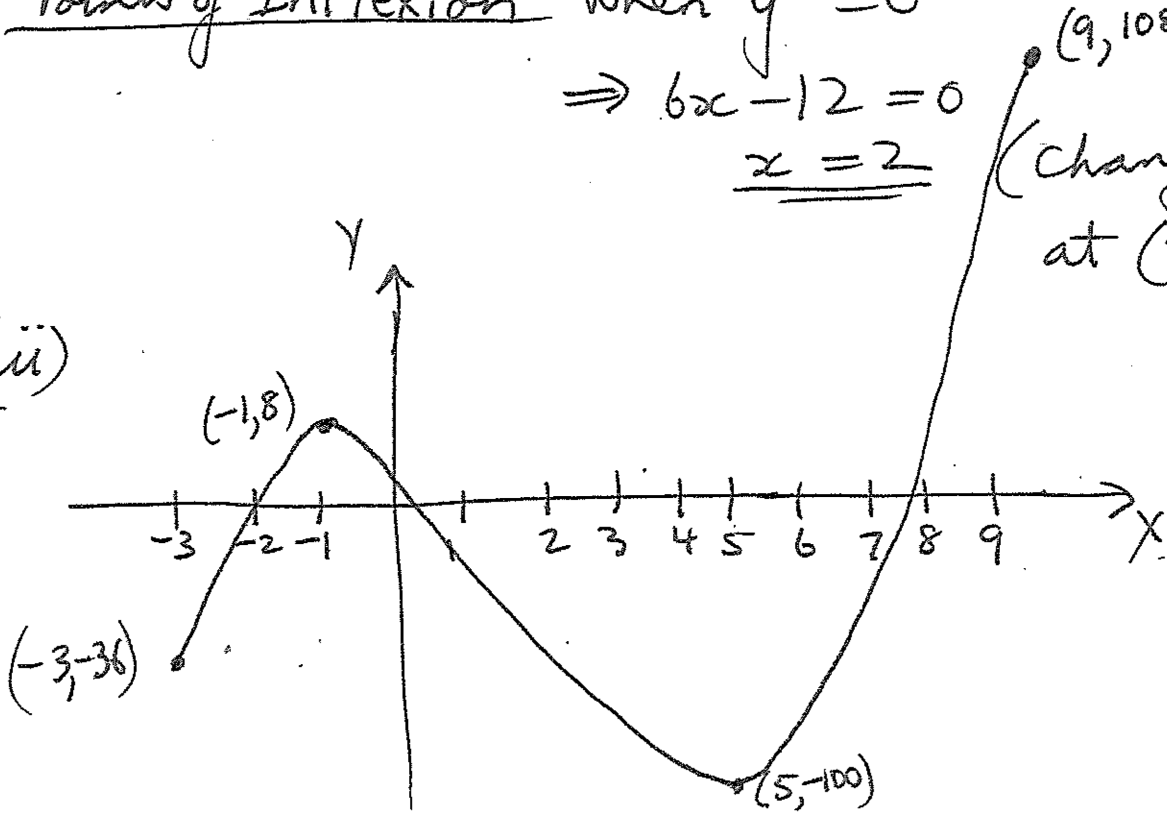
$\Rightarrow 6x - 12 = 0$ ✓

$\underline{x = 2}$

(Change in concavity at $(2, -46)$) ✓

4

(ii)



✓

1

3. (d) (iii) When $x = -3$, $y = -36$] End values
$x = 9$, $y = 108$

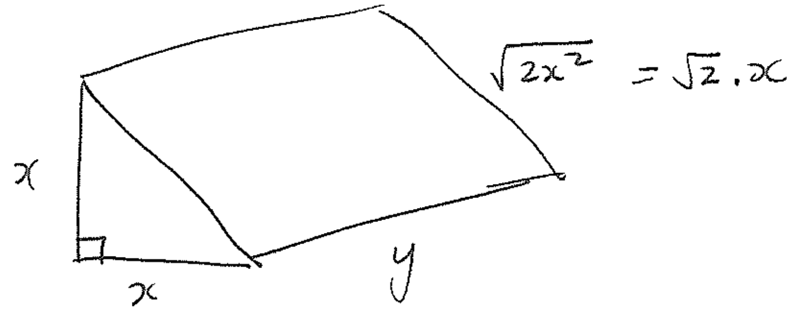
$x = -1$, $y = 8$] Turn. Points
$x = 5$, $y = -100$

$\therefore$ Minimum y value $\underline{\underline{108}}$

Max y value $= \underline{\underline{-100}}$

✓ 1

3(e)



(i) $V = \frac{x^2}{2} . y$ (1)

$S = x^2 + \sqrt{2}\, xy + 2xy$
$= x^2 + (2+\sqrt{2})xy$ (2)

1

(ii) Show $S = x^2 + \frac{2000(2+\sqrt{2})}{x}$

Now $\frac{x^2 y}{2} = 1000$
$\Rightarrow y = \frac{2000}{x^2}$ (3) ✓

Sub (3) in (2)

2

$\Rightarrow S = x^2 + (2+\sqrt{2})x . \frac{2000}{x^2}$
$S = x^2 + \frac{2000(2+\sqrt{2})}{x}$ ✓

(iii) $S' = 2x - \frac{2000(2+\sqrt{2})}{x^2}$ ✓

For turn. pts, $S' = 0 \Rightarrow 2x - \frac{2000(2+\sqrt{2})}{x^2} = 0$
$2x^3 = 2000(2+\sqrt{2})$
$x^3 = 1000(2+\sqrt{2})$ ✓
$x = 15.0578$
$\underline{x = 15.1}$ to 1dp

3(e)(iii)(cont)

$$S'' = 2 + \frac{4000(2+\sqrt{2})}{x^3}$$

$$S''(15.1) = 2 + \frac{4000(2+\sqrt{2})}{(15.1)^3} > 0$$

$\Rightarrow$ min. at $\underline{x = 15.1}$

ie. S.A. is a minimum when $x = 15.1$

3